Outline: Parametric Curves

1. **Velocity, Acceleration, and Speed**

   If $\vec{x}(t) = \big(x(t), y(t)\big)$ is a parametric curve, its **velocity** and **acceleration** are given by the formulas

$$\text{velocity} \;=\; \vec{x}\,'(t) \;=\; \big(x'(t), y'(t)\big) \qquad \text{and} \qquad \text{acceleration} \;=\; \vec{x}\,''(t) \;=\; \big(x''(t), y''(t)\big).$$

   The magnitude of the velocity is called the **speed**:

$$\text{speed} \;=\; \|\text{velocity}\| \;=\; \|\vec{x}\,'(t)\| \;=\; \sqrt{x'(t)^2 + y'(t)^2}.$$

   A curve $\vec{x}(t)$ is called a **unit-speed curve** if its speed is equal to 1 at every point.

2. **Unit Tangent and Unit Normal**

   If $\vec{x}(t) = \big(x(t), y(t)\big)$ is a parametric curve, the **unit tangent vector** $\vec{T}(t)$ is a unit vector in the direction of the velocity:

$$\vec{T}(t) \;=\; \frac{\vec{x}(t)}{\|\vec{x}\,'(t)\|} \;=\; \left(\frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\right).$$

   Note that, if $\vec{x}(t)$ is a unit-speed curve, then the velocity vector $\vec{x}\,'(t)$ and the unit tangent vector $\vec{T}(t)$ are the same.

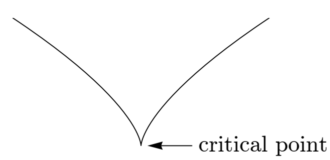
   The **unit normal vector** $\vec{U}(t)$ is the unit vector obtained by turning $\vec{T}(t)$ counterclockwise 90°:

$$\vec{U}(t) \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{T}(t).$$

   That is, if $\vec{T}(t) \;=\; \big(T_1(t), T_2(t)\big)$, then $\vec{U}(t) \;=\; \big(-T_2(t), T_1(t)\big)$.

3. **Critical Points and Regular Curves**

   If $\vec{x}(t)$ is a parametric curve, a **critical point** for $\vec{x}$ is a value of t for which $\vec{x}\,'(t)$ is either undefined or equal to $\vec{0}$. Critical points for parametric curves often correspond to bends or cusps in the curve itself:



   A value of t that is not a critical point is called a **regular point**. A curve $\vec{x}\colon I \to \mathbb{R}^2$ is called **regular** if all of the values of t in I are regular points, i.e. if it has no critical points in its domain.

   Note that the unit tangent vector $\vec{T}(t)$ is only defined at the regular points of a curve. In particular, if $\vec{x}\,'(t) = \vec{0}$, then it isn't possible to define a unit vector in the direction of $\vec{x}\,'(t)$.

4. Arc Length

The **arc length** of a parametric curve $\vec{x}(t) = \big(x(t), y(t)\big)$ for $a \leq t \leq b$ is the integral of its speed:

$$\text{arc length} \;=\; \int_a^b \|\vec{x}\,'(t)\|\,dt \;=\; \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

The **arc length parameter** $s(t)$ represents the arc length starting at a specified point on the curve:

$$s(t) \;=\; \int_a^t \|\vec{x}\,'(\tau)\|\,d\tau \;=\; \int_a^t \sqrt{x'(\tau)^2 + y'(\tau)^2}\,d\tau$$

Note that $s(t)$ is positive for $t > a$, and negative for $t < a$. By the Fundamental Theorem of Calculus, the derivative of s is the speed of the curve:

$$s'(t) \;=\; \text{speed} \;=\; \|\vec{x}\,'(t)\| \;=\; \sqrt{x'(t)^2 + y'(t)^2}.$$

For a unit speed curve, $s'(t) = 1$, and hence $s(t) = t - a$.

5. Reparameterization

If $\mathcal{C}$ is a curve parameterized by $\vec{x}\colon I \to \mathbb{R}^2$, we can **reparameterize** $\mathcal{C}$ by making a substitution of the form $t = f(u)$ in the formula for $\vec{x}(t)$, where f is some invertible function. For example, if

$$\vec{x}(t) \;=\; (\cosh t, \sinh t), \qquad 0 \leq t \leq 9,$$

we can reparameterize this curve by substituting $t = u^2$. This gives us a new parameterization of the same curve:

$$\vec{y}(u) \;=\; \big(\cosh(u^2), \sinh(u^2)\big), \qquad 0 \leq u \leq 3.$$

6. Unit-Speed Parameterizations

We can make any regular curve $\vec{x}(t)$ into a unit-speed curve by using the arc length parameter s. Starting with the formula for $s(t)$, simply solve for s in terms of t and then plug the result into $\vec{x}(t)$. For example, if

$$\vec{x}(t) \;=\; \big(\cos(t^3), \sin(t^3)\big)$$

then the corresponding arc length parameter is $s(t) = t^3$. Solving for t in terms of s gives $t = s^{1/3}$, and plugging this into $\vec{x}(t)$ gives a unit-speed curve:

$$\vec{y}(s) \;=\; \vec{x}(s^{1/3}) \;=\; \big(\cos s, \sin s\big).$$